Ques: Show that the 3D volume element $dx\,dy\,dz$ is not invariant under Lorentz transformation but 4D volume element $dx\,dy\,dz\,dt$ is invariant under lorentz transformation.

Ans:- Suppose an inertial frame S' is moving with velocity v relative to an inertial frame S along +ve x-axis.

From length contraction,

$$dx' = \frac{dx}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}\cdot dx \;,\quad dy' = dy \quad \text{and} \quad dz' = dz$$

From time dialation,

$$dt' = \gamma dt = \frac{dt}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Now 3D volume element in frame $S = dx.dy.dz$

and 3D volume element in frame $S' = dx'.dy'.dz'$.

$$dx'.dy'.dz' = \frac{dx}{\gamma}.dy.dz \quad \text{using length contraction}$$

$$\Rightarrow dx'.dy'.dz' \neq dx.dy.dz$$

Therefore, 3D volume element is variant (not invariant) under lorentz transformation.

Now 4D volume element in frame $S = dx.dy.dz.dt$

4D volume element in frame $S' = dx'.dy'dz'.dt'$

$$dx'.dy'.dz'.dt' = \frac{dx}{\gamma}.dy.dz.\gamma dt \quad \text{using length contraction and time dialation.}$$

$$\Rightarrow dx'.dy'.dz'dt' = dx.dy.dz.dt.$$

Thus 4D volume element is invariant (not variant) under lorentz transformation.

Ques:- Show that 3D coordinate (the space Interval) $x^2+y^2+z^2$ is not invariant under lorentz transformation but 4D coordinates (the space-time interval) $x^2+y^2+z^2-c^2t^2$ is invariant under Lorentz transformation.

Ans:- From Lorentz transformation equation,

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}\;,\quad y' = y\;,\quad z' = z \quad \text{and} \quad t' = \frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}.$$

space interval in frame $S = x^2+y^2+z^2$

space interval in frame $S' = x'^2+y'^2+z'^2$

Now $x'^2+y'^2+z'^2 = \dfrac{(x-vt)^2}{1-\frac{v^2}{c^2}} + y^2+z^2$ Using Lorentz transformation

$$\Rightarrow x'^2+y'^2+z'^2 \neq x^2+y^2+z^2$$

Thus space interval (3D coordinates) is variant (not invariant) under Lorentz transformation.

Again space-time interval in frame $S = x^2+y^2+z^2-c^2t^2$

and space-time interval in frame $S' = x'^2+y'^2+z'^2-c^2t'^2$

Now $x'^2+y'^2+z'^2-c^2t'^2 = \dfrac{(x-vt)^2}{1-\frac{v^2}{c^2}} + y^2+z^2 - \dfrac{c^2\left(t-\frac{v}{c^2}x\right)^2}{1-\frac{v^2}{c^2}}$

(using Lorentz transformation)

$$\Rightarrow x'^2+y'^2+z'^2-c^2t'^2 = \frac{1}{1-\frac{v^2}{c^2}}\left(x^2+v^2t^2-\cancel{2xvt}-c^2t^2-\frac{v^2x^2}{c^2}+\cancel{2xvt}\right) + y^2+z^2$$

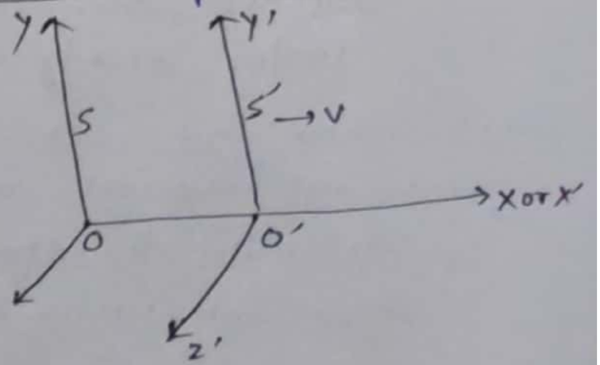
$$= \frac{1}{1-\frac{v^2}{c^2}}\left[x^2\left(1-\frac{v^2}{c^2}\right) - c^2t^2\left(1-\frac{v^2}{c^2}\right)\right] + y^2+z^2$$

$$\Rightarrow x'^2+y'^2+z'^2-c^2t'^2 = x^2+y^2+z^2-c^2t^2$$

Thus space-time interval (4D coordinate) is invariant (not variant) under Lorentz transformation.

* Lorentz transformation of Differential operator

Let us consider two inertial frames S and S' in which the frame S' is moving with velocity v relative to the frame S along +ve x axis.



From Lorentz transformation equation

$$x' = \gamma(x-vt), \quad y'=y, \quad z'=z \quad \text{and} \quad t' = \gamma\left(t-\frac{v}{c^2}x\right) \quad \text{——} (A)$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-v^2/c^2}}, \quad \beta = \dfrac{v}{c}$

Functional relations given by the Lorentz transformations may be written as

$$x' = f(x,t), \quad y' = f(y), \quad z' = f(z) \text{ and } t' = f(x,t)$$

Now $$\frac{\delta}{\delta x'} = \frac{\delta}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial x'} \quad \text{—— (i)}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial y'} \quad \text{—— (ii)}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial z'} \quad \text{—— (iii)}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial t'} + \frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial t'} \quad \text{—— (iv)}$$

From inverse Lorentz transformation

$$x = \gamma(x'+vt'), \quad y = y', \quad z = z' \text{ and } t = \gamma\left(t' + \frac{v}{c^2}x'\right) \quad \text{—— (B)}$$

On differentiating equ (B) partially, we get.

$$\frac{\partial x}{\partial x'} = \gamma\left(\frac{\delta x'}{\partial x'} + 0\right) \Rightarrow \frac{\partial x}{\partial x'} = \gamma\cdot 1 \Rightarrow \frac{\partial x}{\partial x'} = \gamma \quad \text{—— (v)}$$

$$\frac{\partial x}{\partial t'} = \gamma\left(0 + v\frac{\partial t'}{\partial t'}\right) = \gamma\cdot v\cdot 1 \Rightarrow \frac{\partial x}{\partial t'} = \gamma v \quad \text{—— (vi)}$$

$$\frac{\partial y}{\partial y'} = \frac{\partial y'}{\partial y'} = 1 \Rightarrow \frac{\partial y}{\partial y'} = 1 \quad \text{—— (vii)}$$

$$\frac{\partial z}{\partial z'} = \frac{\partial z'}{\partial z'} = 1 \Rightarrow \frac{\partial z}{\partial z'} = 1 \quad \text{—— (viii)}$$

$$\frac{\partial t}{\partial t'} = \gamma\left(\frac{\delta t'}{\delta t} + 0\right) = \gamma\cdot 1 \Rightarrow \frac{\partial t}{\partial t'} = \gamma \quad \text{—— (ix)}$$

and $$\frac{\partial t}{\partial x'} = \gamma\left(0 + \frac{v}{c^2}\cdot\frac{\partial x'}{\partial x'}\right) = \gamma\frac{v}{c^2}\cdot 1 \Rightarrow \frac{\partial t}{\partial x'} = \gamma\cdot\frac{v}{c^2} \quad \text{—— (x)}$$

Using equ (v) and (x) in equ (1), we get

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\cdot\gamma + \frac{\partial}{\partial t}\cdot\gamma\cdot\frac{v}{c^2} \Rightarrow \frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right) \quad \text{—— (C-1)}$$

Using equ (vii) in equ (ii), we get $$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\cdot 1 \Rightarrow \frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \quad \text{—— (C-2)}$$

Using equ (viii) in equ (iii), we get $$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}\cdot 1 \Rightarrow \frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \quad \text{—— (C-3)}$$

Using equ^n ⑤ (vi) and (ix) in equ^n (iv), we get

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\cdot \gamma v + \frac{\partial}{\partial t}\cdot \gamma \Rightarrow \frac{\partial}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \quad \text{(C-4)}$$

Thus Lorentz transformation of differential operator will be

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\cdot\frac{\partial}{\partial t}\right) \\ \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z} \\ \text{and } \frac{\partial}{\partial t'} &= \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \end{aligned}\right\} \quad \text{Ⓒ}$$

Inverse Lorentz transformation of differential operator can be obtained by interchanging primed and unprimed quantities and by putting $-v$ in place of v in equ^n Ⓒ because the frame s is moving with velocity v relative to the frame s' along $-ve$ x-axis.

Thus Inverse Lorentz transformation of differential operator will be

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\cdot\frac{\partial}{\partial t'}\right) \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} \\ \text{and } \frac{\partial}{\partial t} &= \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right) \end{aligned}\right\} \quad \text{Ⓓ}$$

Here c is not taken as unprimed (c') because $c = c'$ is, velocity of light c remains unchanged in all inertial frame according to principle of special theory of relativity.